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SUMMARY

A theoretical study was made to determine the utility of a technique for the synthesis of constant linear dynamical systems with a bang-bang controller. The technique employs linear switching logic and the premise that the use of time-dependent gains eliminates endpoints. Conditions for asymptotic stability were obtained from consideration of Liapunov functions.

As part of the investigation, the technique was applied in an analog computer simulation of an idealized attitude-control system for spinning space vehicles. The system was idealized in the sense that the equations of motion were linearized and time lags in control-system components were neglected. The analog simulation demonstrated the stability and flexibility of the closed-loop control logic under a variety of conditions, and no endpoints were encountered.

INTRODUCTION

In the last few years the theory of optimal control processes has received much attention. (For example, see refs. 1 to 10.) Much of this attention has been focused on the problem of time optimal control of linear dynamical systems with limited control inputs. The theoretical solution of this problem is well-known and example applications of the theory have appeared in the literature. (For example, see refs. 11 and 12.) However, the practical problems of mechanizing closed-loop time optimal systems have not been solved. This difficulty stems from the fact that the time optimal control logic requires a complete knowledge of the state of the system, and a relatively sophisticated computer in the control loop. With present-day computing equipment, these requirements rule out the feasibility of time optimal control in some applications to small spacecraft. It may be impractical, or perhaps impossible, to determine completely the state of the system; moreover, limitations on weight, space, reliability, complexity, and so forth, may prohibit the use of sophisticated computers in the control loop. Therefore, it seems reasonable to investigate control logics which yield control laws with the same analytical structure as the optimal control, but which appear to be relatively easy to mechanize.

The purpose of this investigation was to study the applicability of linear switching logic to the regulator problem where the plant is a linear dynamical system with constant coefficients. Since constant gains in the control loop lead to endpoints (see ref. 13), attention was centered on the use of time-dependent gains.

The analysis was conducted by determining conditions for asymptotic stability from properties of Liapunov functions. Once these conditions were developed, a heuristic synthesis technique was postulated. This technique was successfully applied in an analog simulation of an idealized attitude control system for spinning space vehicles. The system was idealized in that the equations of motion were linearized and time lags in control-system components were neglected. Results of the analog simulation are included in the report.

SYMBOLS

$[A]$	$n \times n$ matrix of constant coefficients determined by dynamics of system
$\vec{a}$	constant n-vector determined by dynamics of system
$[B]$	constant positive definite $n \times n$ matrix, defined as solution of equation (7)
$\hat{b}$	particular form of optimum gain vector defined by equation (6)
$\vec{b}$	n-dimensional gain vector
$[C]$	constant positive definite $n \times n$ matrix used in performance index (eq. (4))
$\vec{e}$	unit vector in direction of free gyro spin axis
I_X,I_Y,I_Z	moments of inertia of vehicle about principal vehicle X-, Y-, and Z-axes, respectively, slug-ft^2
I	transverse moment of inertia when $I_Y = I_Z$, slug-ft^2
$\vec{r},\vec{j},\vec{k}$	unit vectors along principal X-, Y-, and Z-axes, respectively
J	maximum available control acceleration, radians/sec^2
M_Y,M_Z	external pitching and yawing moments, respectively, in principal vehicle-axis coordinate system, ft-lb
$N_\rho(\vec{b})$	neighborhood of point $\vec{b}$ with radius ρ

n	integer
$\vec{0}$	null vector
p,q,r	angular velocities about principal X-, Y-, and Z-axes, respectively, radians/sec
p_o	positive constant spin rate of vehicle about X-axis, radians/sec
s	Laplace transform variable
t	time, sec
u	scalar control parameter
$\hat{u}$	particular form of optimal control law (see eqs. (5) and (6))
V	Liapunov function defined by equation (8)
X,Y,Z	principal vehicle-axis coordinates
$\vec{x}$	n-dimensional state vector

$$\|\vec{x}\|^2 = \sum_{i=1}^{n} x_i^2$$

α,β	gimbal angles of free gyro measured relative to spinning principal body axes, radians
λ	eigenvalue of matrix, defined by equation (A19)
ρ	radius of neighborhood, defined by equation (15)
$\vec{\Omega}$	vector angular velocity of X-, Y-, and Z-axis system, radians/sec
$\sigma(\vec{x},t)$	switching function
$\mathrm{sgn}\,[\ ]$	algebraic sign of term within bracket
ω	frequency defined by equation (A3), radians/sec
$\det(\ )$	determinant of quantity in parentheses
$\|\ \|$	Euclidean norm

$[\ ]$ square matrix

$|\ |$ absolute expression

Subscript o denotes initial value. An integer subscript denotes a component of a vector. Dots over symbols denote differentiation with respect to time. An arrow above a symbol denotes a vector. An asterisk denotes the transpose of a matrix and of a vector.

ANALYSIS

Problem Statement

The problem considered in this investigation is stated as follows. Let a system be described by a set of n differential equations which can be written in vector-matrix form as

$$\dot{\vec{x}} = [A]\vec{x} + u\vec{a} \qquad \left(\vec{x}(0) = \vec{x}_o\right) \tag{1}$$

where $\vec{x}$ is the n-dimensional state vector, $[A]$ is a constant $n \times n$ matrix, $\vec{a}$ is a constant n-vector, and u, the scalar control parameter, is a function of $\vec{x}$ and t. For a bang-bang control system, the control function u has the form

$$u = \operatorname{sgn}\left[\sigma(\vec{x},t)\right] \tag{2}$$

where $\operatorname{sgn}\left[\sigma(\vec{x},t)\right]$ means the algebraic sign of $\sigma(\vec{x},t)$ so that u is restricted to values of ± 1. The function $\sigma(\vec{x},t)$ is generally called the switching function.

Assume that the system is completely controllable in the sense of reference 7 and that $\vec{x}_o \neq 0$, and find a $u(\vec{x},t)$ (provided one exists) which makes $\vec{x}(t) \to 0$ as $t \to \infty$.

Formulation of Solution

Method of formulation.- The solution presented herein is formulated by first examining some of the properties of a linear control system and then considering how the linear control law might be modified to produce a stable nonlinear (bang-bang) control law. The conditions for stability are derived from considerations of Liapunov functions.

A linear control law.- A linear control law is the solution to the following problem. Given the completely controllable system

$$\dot{\vec{x}} = [A]\vec{x} + u\vec{a} \qquad \left(\vec{x}(0) = \vec{x}_0 \neq 0\right) \tag{3}$$

find the control law, say $\hat{u}(\vec{x},t)$, which minimizes the integral

$$\frac{1}{2}\int_0^{+\infty}\left(\vec{x}\cdot[C]\vec{x} + u^2\right)dt \tag{4}$$

where $[C] = [C]^*$ is a given positive definite matrix.

From results given in reference 14, $\hat{u}$ is given by

$$\hat{u} = \hat{\vec{b}}\cdot\vec{x} \tag{5}$$

with the constant n-vector $\hat{\vec{b}}$ being given by

$$\hat{\vec{b}} = -[B]\vec{a} \tag{6}$$

where the symmetric positive definite matrix $[B]$ satisfies

$$[B][A] + [A]^*[B] - [B]\vec{a}\vec{a}^*[B] = -[C] \tag{7}$$

Stability properties of linear control system.- If a positive definite Liapunov function V is defined by

$$V = \vec{x}\cdot[B]\vec{x} \tag{8}$$

it has the negative definite time derivative given by

$$\dot{V} = \vec{x}\cdot\left([B][A] + [A]^*[B] - 2[B]\vec{a}\vec{a}^*[B]\right)\vec{x} = -\left(\vec{x}\cdot[C]\vec{x} + \hat{u}^2\right) \tag{9}$$

from which it follows (as in ref. 14) that the system (eq. (3)) with control $\hat{u}$ is exponentially asymptotically stable.

Consider the following question: How could the gain vector $\hat{\vec{b}}$ be varied without disrupting the stability of the linear control system? Estimates of the variation can be calculated by setting

$$u = \left(\hat{\vec{b}} + \overrightarrow{\Delta b}\right)\cdot\vec{x} \tag{10}$$

calculating the time derivative of V, which leads to

$$\dot{V} = -\left(\vec{x} \cdot [C]\vec{x} + \hat{u}^2\right) - 2\left(\hat{\vec{b}} \cdot \vec{x}\right)\left(\overrightarrow{\Delta b} \cdot \vec{x}\right) \tag{11}$$

and determining conditions on $\overrightarrow{\Delta b}$ such that $\dot{V} < 0$.

The approach taken herein is to consider the following two cases: (1) $\overrightarrow{\Delta b}$ colinear with $\hat{\vec{b}}$ and (2) $\overrightarrow{\Delta b}$ not colinear with $\hat{\vec{b}}$.

Case (1): If $\overrightarrow{\Delta b}$ is colinear with $\hat{\vec{b}}$, let

$$\overrightarrow{\Delta b} = \theta\hat{\vec{b}} \tag{12}$$

where θ is a scalar parameter. Then equation (11) can be rewritten as

$$\dot{V} = -\left(\vec{x} \cdot [C]\vec{x} + \hat{u}^2\right) - 2\theta\left(\hat{\vec{b}} \cdot \vec{x}\right)^2 \tag{13}$$

from which it is immediately obvious that $\dot{V} < 0$ for all $\theta \geqq 0$. Furthermore, it can be shown that $\dot{V} < 0$ if

$$\frac{-\mu_1}{2\left\|\hat{\vec{b}}\right\|^2} < \theta \leqq \infty \tag{14}$$

where $\mu_1 > 0$ is the smallest eigenvalue of $[C]$.

Case (2): If $\overrightarrow{\Delta b}$ is not colinear with $\hat{\vec{b}}$, it can be shown that $\dot{V}$ (given by eq. (11)) will be negative if

$$\left\|\overrightarrow{\Delta b}\right\| < \frac{\mu_1}{2\left\|\hat{\vec{b}}\right\|} = \rho \tag{15}$$

where the middle expression defines ρ.

This result has the geometrical interpretation represented in figure 1. Inequality (15) means that there is a neighborhood of radius ρ surrounding the point defined by the vector $\hat{\vec{b}}$ and every gain vector defining a point in this neighborhood is a stable gain vector for the linear control system. In figure 1 the neighborhood is represented by the interior of the circle with radius ρ and center at the point defined by vector $\hat{\vec{b}}$. This neighborhood is denoted by $N_\rho\left(\hat{\vec{b}}\right)$.

It is interesting to note that the estimate on the lower bound of θ is in perfect agreement with inequality (15). This agreement can be seen as

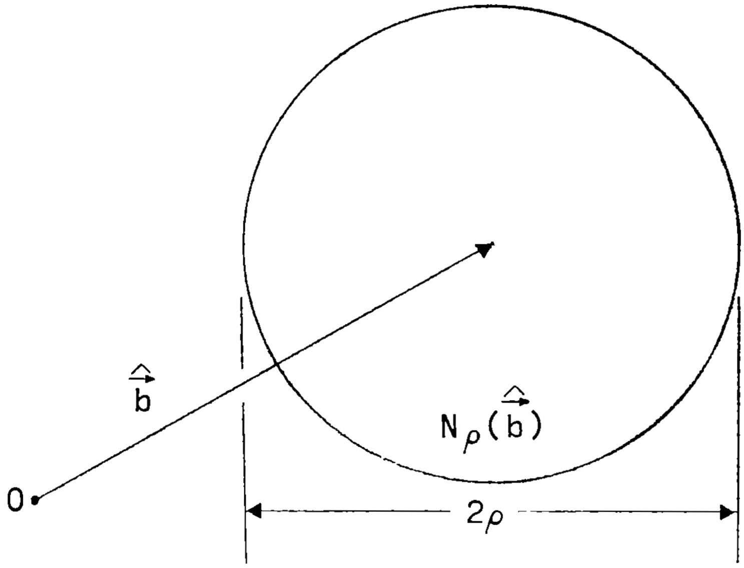


Figure 1.- Geometrical representation of stable gain region.

follows. In equation (12) replace θ by the lower bound estimate given in inequality (14) to get

$$\overrightarrow{\Delta b} = \frac{-\mu_1 \hat{\vec{b}}}{2\left\|\hat{\vec{b}}\right\|^2} = \frac{-\mu_1}{2\left\|\hat{\vec{b}}\right\|} \cdot \frac{\hat{\vec{b}}}{\left\|\hat{\vec{b}}\right\|} \equiv \frac{-\rho\hat{\vec{b}}}{\left\|\hat{\vec{b}}\right\|} \tag{16}$$

which says that $\overrightarrow{\Delta b}$ is in the opposite direction of $\hat{\vec{b}}$ and has magnitude ρ. From the geometry of figure 1 it is clear that the estimates agree.

These properties of the linear control law having been noted, attention is now turned to the nonlinear (bang-bang) case.

A bang-bang control law.- If the switching function $\sigma(\vec{x},t)$ in equation (2) is given by $\hat{\vec{b}} \cdot \vec{x}$, u can be written as

$$u = \mathrm{sgn}\left[\hat{\vec{b}} \cdot \vec{x}\right] = \frac{\hat{\vec{b}} \cdot \vec{x}}{\left|\hat{\vec{b}} \cdot \vec{x}\right|} \tag{17}$$

where u is understood to be zero when $\hat{\vec{b}} \cdot \vec{x} = 0$. Then if equation (17) is used as the control law in system (1), equation (1) can be rewritten as

$$\dot{\vec{x}} = \left[A\right]\vec{x} + \vec{a}\,\frac{\hat{\vec{b}} \cdot \vec{x}}{\left|\hat{\vec{b}} \cdot \vec{x}\right|} \tag{18}$$

It is well-known (see ref. 13) that this control law leads to endpoints; that is, it is impossible to use $\hat{\vec{b}} \cdot \vec{x}$ as the switching function (where $\hat{\vec{b}}$ is a constant n-vector) without having a region in which the control will begin to chatter if the trajectory (solution of eq. (18)) hits the hyperplane $\hat{\vec{b}} \cdot \vec{x} = 0$ with $\vec{x} \neq 0$. This facet of the problem will be treated after considering the stability of the system (eq. (18)).

The stability of the system described by equation (18) is readily examined by defining a scalar parameter θ such that

$$1 + \theta \equiv \frac{1}{\left|\hat{\vec{b}} \cdot \vec{x}\right|} \tag{19}$$

Then, with a straightforward substitution, equation (18) can be rewritten as

$$\dot{\vec{x}} = [A]\vec{x} + \vec{a}\left[\left(\hat{\vec{b}} + \theta\hat{\vec{b}}\right)\right] \cdot \vec{x} \tag{20}$$

which shows that this bang-bang control law can be regarded as a linear control law with a colinear variation of the gain vector, that is,

$$u = \left(\hat{\vec{b}} + \overrightarrow{\Delta b}\right) \cdot \vec{x} = \left(\hat{\vec{b}} + \theta\hat{\vec{b}}\right) \cdot \vec{x} \tag{21}$$

Since it is known (from the stability properties of the linear system) that the system is stable for $\theta > \dfrac{-\mu_1}{2\left\|\hat{\vec{b}}\right\|^2}$, equation (19) can be solved for θ and the result used in the left-hand term of inequality (14) to determine stability conditions. This procedure leads to

$$\frac{1}{\left|\hat{\vec{b}} \cdot \vec{x}\right|} - 1 > \frac{-\mu_1}{2\left\|\hat{\vec{b}}\right\|^2} \tag{22}$$

or

$$\frac{1}{\left|\hat{\vec{b}} \cdot \vec{x}\right|} > 1 - \frac{\mu_1}{2\left\|\hat{\vec{b}}\right\|^2} \tag{23}$$

as a sufficient condition for stability. It is clear that inequality (23) must hold if the right-hand side is negative; however, this is true only if the system is asymptotically stable without control. To prove this statement, proceed as follows: Suppose the right-hand side of inequality (23) is negative. Then

$$1 - \frac{\mu_1}{2\left\|\hat{\vec{b}}\right\|^2} < 0 \tag{24}$$

which can be rewritten as

$$\left\|\hat{\vec{b}}\right\| < \frac{\mu_1}{2\left\|\hat{\vec{b}}\right\|} \equiv \rho \tag{25}$$

From the geometry of figure (1) it is easy to see that inequality (25) holds only if $N_\rho\left(\hat{\vec{b}}\right)$ contains the origin, which in turn is true only if the uncontrolled system is asymptotically stable. This relation follows from the fact that the use of any $\vec{b}$ in $N_\rho\left(\hat{\vec{b}}\right)$ makes the linear system asymptotically stable; therefore, if $\vec{0}$ is in $N_\rho\left(\hat{\vec{b}}\right)$, the system must be asymptotically stable without control.

Suppose that the right-hand side of inequality (23) is positive but clearly less than unity. The inequality then would hold if

$$\frac{1}{\left|\hat{\vec{b}} \cdot \vec{x}\right|} \geqq \frac{1}{\left\|\hat{\vec{b}}\right\| \cdot \left\|\vec{x}\right\|} > 1 \tag{26}$$

from which a sufficient condition for $\dot{V} < 0$ is

$$\left\|\vec{x}\right\| < \frac{1}{\left\|\hat{\vec{b}}\right\|} \tag{27}$$

The significance of this inequality is explained by considering the fact that the choice of $\left[C\right] = \left[C\right]^* > 0$ is arbitrary so that $\hat{\vec{b}}$ is any gain vector, of the form

$$\hat{\vec{b}} = -\left[B\right]\vec{a} \qquad \left(\left[B\right] = \left[B\right]^* > 0\right) \tag{28}$$

which is a stable gain vector for the linear control system. Therefore, the minimum $\left\|\hat{\vec{b}}\right\|$ of this form may be regarded as a measure of the minimum amount of "muscle" required to stabilize the linear control system. If M is defined as the minimum $\left\|\hat{\vec{b}}\right\|$ considering all possible positive matrices in equation (4), that is,

$$M \triangleq \min_{\left[C\right]} \left\{\left\|\hat{\vec{b}}\right\|\right\} \tag{29}$$

then inequality (27) can be rewritten as

$$\left\|\vec{x}\right\| < \frac{1}{M} \tag{30}$$

The following conclusions can then be drawn relative to the stability of the bang-bang control system (eq. (18)) prior to an endpoint.

(1) If the uncontrolled system is exponentially asymptotically stable ($M = 0$), the controlled system will be asymptotically stable.

(2) If the uncontrolled system is neutrally stable in the sense that $M > 0$ may be arbitrarily small (from inequality (30)), the controlled system will be asymptotically stable for all finite $\|\vec{x}\|$.

(3) If the uncontrolled system is unstable in the sense that $M > 0$ must be finite, the controlled system will be stable if $\|\vec{x}\| < M^{-1}$.

From physical considerations it is clear that when the uncontrolled system is unstable ($M > 0$) and the muscle is limited (as in the case of bang-bang control), the system must remain in a neighborhood of the origin (estimated by inequality (30)) where the muscle can override the instabilities of the system.

The endpoint problem.- In reference 13 it is proved that every state vector $\vec{x}$ which satisfies

$$\hat{\vec{b}} \cdot \vec{x} = 0 \tag{31}$$

and

$$-\vec{a} \cdot [B]\vec{a} = \hat{\vec{b}} \cdot \vec{a} < \vec{x} \cdot [A]^*\hat{\vec{b}} < -\hat{\vec{b}} \cdot \vec{a} = \vec{a} \cdot [B]\vec{a} \tag{32}$$

is an endpoint. That is to say, that when the trajectory (solution of eq. (18)) hits the hyperplane $\hat{\vec{b}} \cdot \vec{x} = 0$ at one of these points, the state vector $\vec{x}$ gets trapped in the hyperplane and the control begins to chatter since $\hat{\vec{b}} \cdot \vec{x}$ is the control switching function.

Since the problem lies in the fact that $\vec{x}$ cannot get out of the hyperplane $\left(\hat{\vec{b}} \cdot \vec{x} = 0\right)$, it seemed reasonable to ask: Can this problem be alleviated by moving the hyperplane away from the state vector? If it is assumed that the answer to this question is yes, the groundwork has been laid for the following heuristic synthesis technique.

Heuristic synthesis technique.- After the stability conditions have been investigated and a solution to the endpoint problem assumed, the heuristic synthesis technique consists of the following steps:

Given the system

$$\dot{\vec{x}} = [A]\vec{x} + u\vec{a} \qquad \left(\vec{x}(0) = \vec{x}_0\right)$$

subject to the constraints $|u| \leqq 1$ and

$$\det\left(\vec{a},\ [A]\vec{a},\ [A]^2\vec{a},\ .\ .\ .\ [A]^{n-1}\vec{a}\right) \neq 0$$

select a $\hat{\vec{b}}$ of the form $-[B]\vec{a}$ such that all the roots of

$$\det\left([A] + \vec{a}\hat{\vec{b}}^* - \lambda[I]\right) = 0$$

have negative real parts. Let $\vec{b}(t)$ (of the form $-[B]\vec{a}$ for all $t \geqq 0$) be given by

$$\vec{b}(t) = \hat{\vec{b}} + \overrightarrow{\Delta b}(t)$$

(where $\hat{\vec{b}}$ is a constant vector and $\|\Delta b(t)\|$ is sufficiently small for all $t \geqq 0$). Then a stable bang-bang control law is obtained by setting

$$u(\vec{x};t) = \operatorname{sgn}\left[\vec{b}(t) \cdot \vec{x}\right]$$

To generate some feeling for its utility, this synthesis technique was applied in an analog simulation of an idealized attitude control system for spinning space vehicles.

APPLICATION OF SYNTHESIS TECHNIQUE TO ATTITUDE CONTROL OF SPINNING SPACE VEHICLE

Description of Control System

Figure 2 represents the system components. The X,Y,Z axes are principal vehicle axes. The vehicle spins about the X-axis to provide basic gyroscopic stability. The control system consists of a free gyro, one rate gyro, pitch jets, and a computer. The free gyro is set so that its spin vector and total angular momentum vector are coincident with some reference direction in inertial space. Since the gyro is free (frictionless gimbal bearings), it will maintain this attitude. If the X-axis is parallel to the spin vector of the free gyro, and if at the same time $q = r = 0$ (pitch and yaw rates are zero) and no torques are acting on the vehicle, the vehicle will have the desired reference attitude and tend to maintain it. If the X-axis is not parallel to the free gyro spin vector, the misalinement will be sensed by a pickoff device sensing the gimbal angle β which is detailed in figure 3. Thus the free gyro and the rate gyro, which senses q, supply the computer with the information needed to determine control torques. The control torques are supplied by the pitch jets in accordance with the output of the computer. A block diagram of the system is given in figure 4.

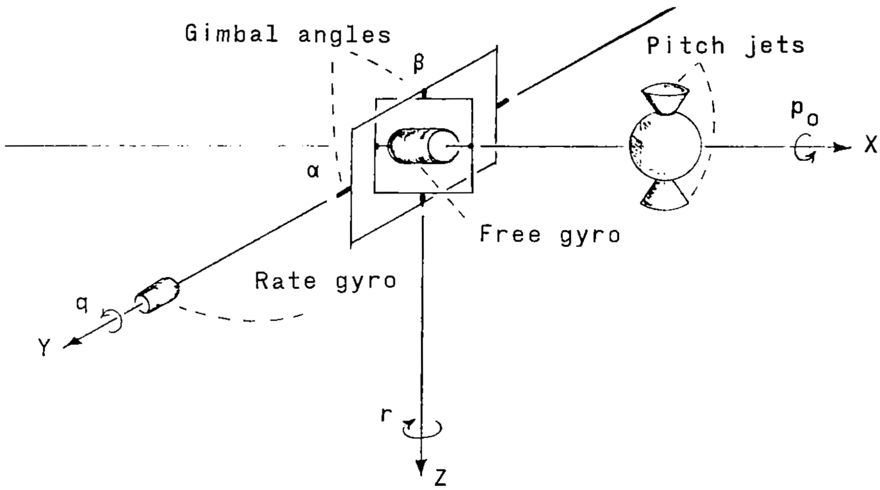


Figure 2.- Illustration of control system for spinning space vehicle. X, Y, and Z indicate the principal vehicle-fixed axes.

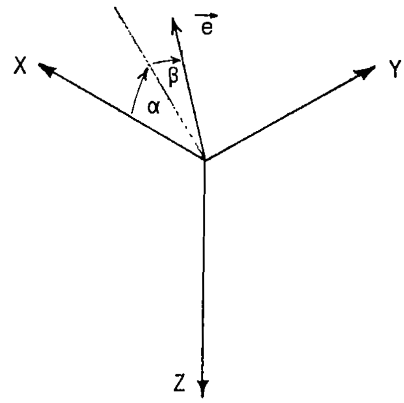


Figure 3.- Detail of gimbal angles α and β. The unit vector $\vec{e}$ has the direction of the free gyro spin axis with orientation given by α and β, measured relative to the X-, Y-, and Z-axis system.

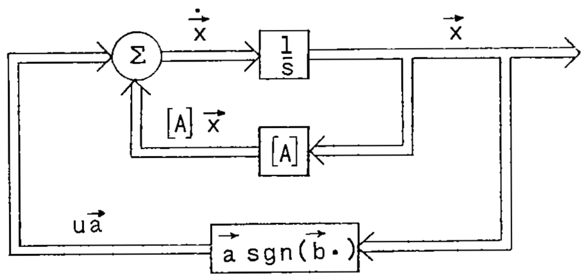


Figure 4.- Block diagram of closed-loop bang-bang control system.

Dynamic and Control Logic Equations

The equations which were used to describe this system are developed in the appendix. The resulting equations and control relations in explicit form are as follows:

$$
\left.\begin{aligned}
\dot{q} &= \omega r + Ju \\
\dot{r} &= -\omega q \\
\dot{\alpha} &= p_o\beta - q \\
\dot{\beta} &= -p_o\alpha - r
\end{aligned}\right\} \tag{33}
$$

$$u(\vec{x};t) = \operatorname{sgn}\left[\vec{b}(t) \cdot \vec{x}\right] \tag{34}$$

$$\vec{b}(t) = \begin{bmatrix} \hat{b}_1 - \cos 2\omega t \\ 0 \\ 0 \\ \hat{b}_4 \end{bmatrix} \tag{35}$$

The frequency 2ω used in $\overrightarrow{\Delta b}(t)$ was chosen arbitrarily, and the results of the analog study verified the fact that system response is not critically dependent upon this quantity.

Analog Simulation of Control System

An analog computer simulation of a representative system was mechanized by using equations (33), (34), (35), and the following values:

I_X/I	0.2
p_O, radians/sec	25
ω, radians/sec	20
J, radians/sec/sec	1.0

Values for $\hat{b}_1$ and $\hat{b}_4$ were determined simply by varying these gains until the system response seemed to be the best obtainable. The simulation was then used to study system performance in a variety of circumstances. Consideration was given to the effects of varied initial conditions, magnitude of control torque, presence of external torques, variation of spin rate, and combinations of these.

RESULTS AND DISCUSSION

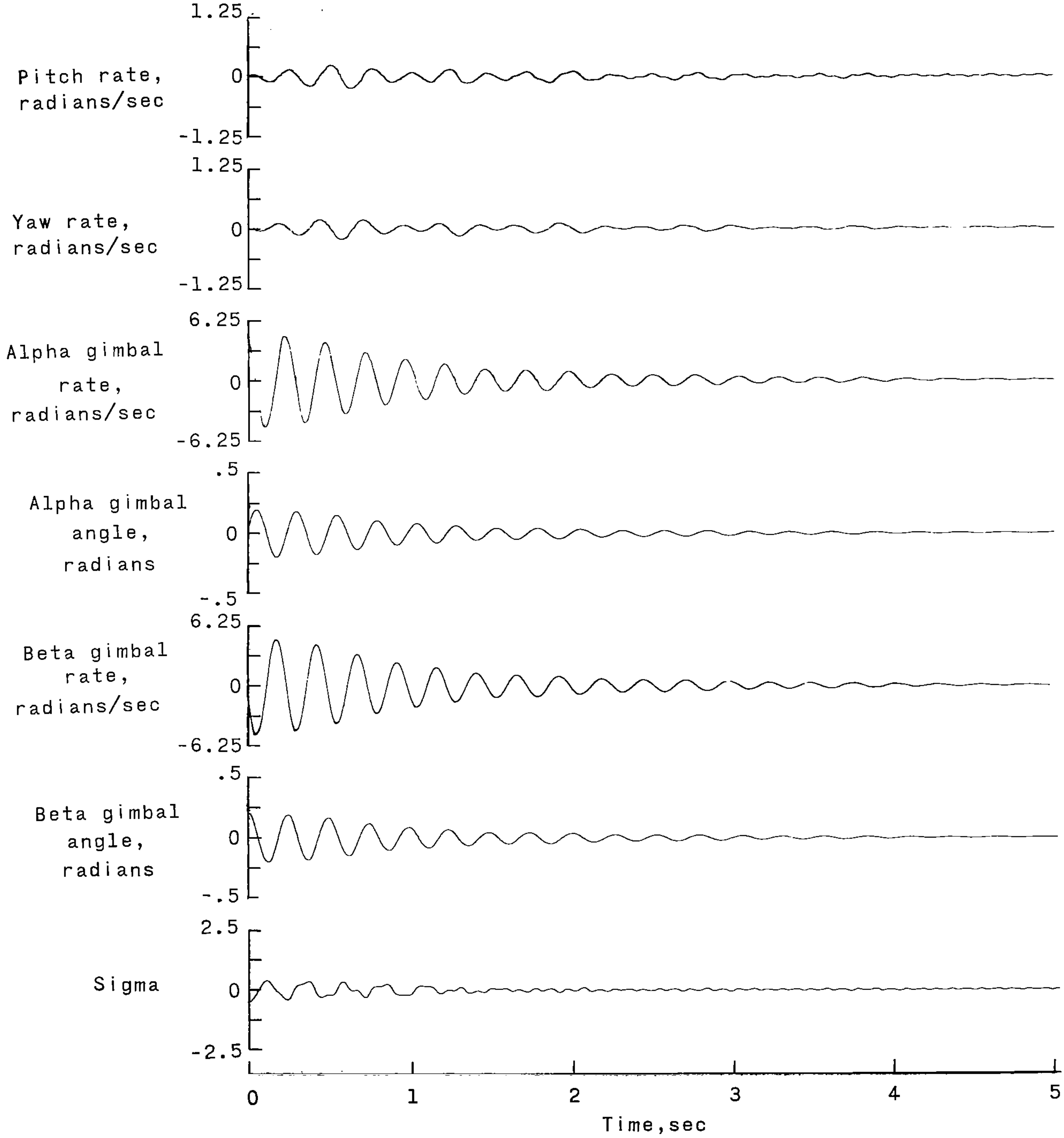
The analog simulation of the spinning body control system was used to evaluate system performance for a variety of conditions of interest. The results of this investigation are summarized by considering figures 5 to 11.

The first objective of the analog study was to determine the best values for the gains $\hat{b}_1$ and $\hat{b}_4$ and to check the sharpness of the stability limit given in the appendix $\left(0 < b_4 < -\frac{b_1 I_X p_o}{I}\right)$. It was found that for high values of $\left|\hat{b}_1\right|$, limit cycle difficulties exist. Actually, these were not true limit cycles, but as the system approached the origin, damping became very light. As the value of $\left|\hat{b}_1\right|$ was decreased $\left(\hat{b}_4\right.$ being held constant$\left.\right)$ toward the stability limit, the overall damping decreased but damping close to the origin improved. The system remained stable across the limit but instability was encountered when the limit was grossly violated. As expected, higher values of the ratio $\left|\hat{b}_4/\hat{b}_1\right|$ produced better damping of the angles $\left(\hat{b}_4\right.$ is essentially the angular error gain$\left.\right)$. However, the rate damping became relatively poor for high values of this ratio. The inequality $0 < b_4 < -\frac{b_1 I_X p_o}{I}$ proved to be a conservative stability limit. Consideration of the data led to the selection of the values $\hat{b}_1 = -1.5$ and $\hat{b}_4 = 2.5$ for further study. It was felt that these values gave the best compromise in evaluating all the factors.

The effect of control torque magnitude was investigated and, as expected, the system responded faster as control torque was increased. The value $J = 1.0$ was selected arbitrarily for further study. With these parameter values, figure 5 shows typical system responses to arbitrary initial conditions. Note that the response to all conditions is smooth and the damping strong, with no limit cycle difficulties.

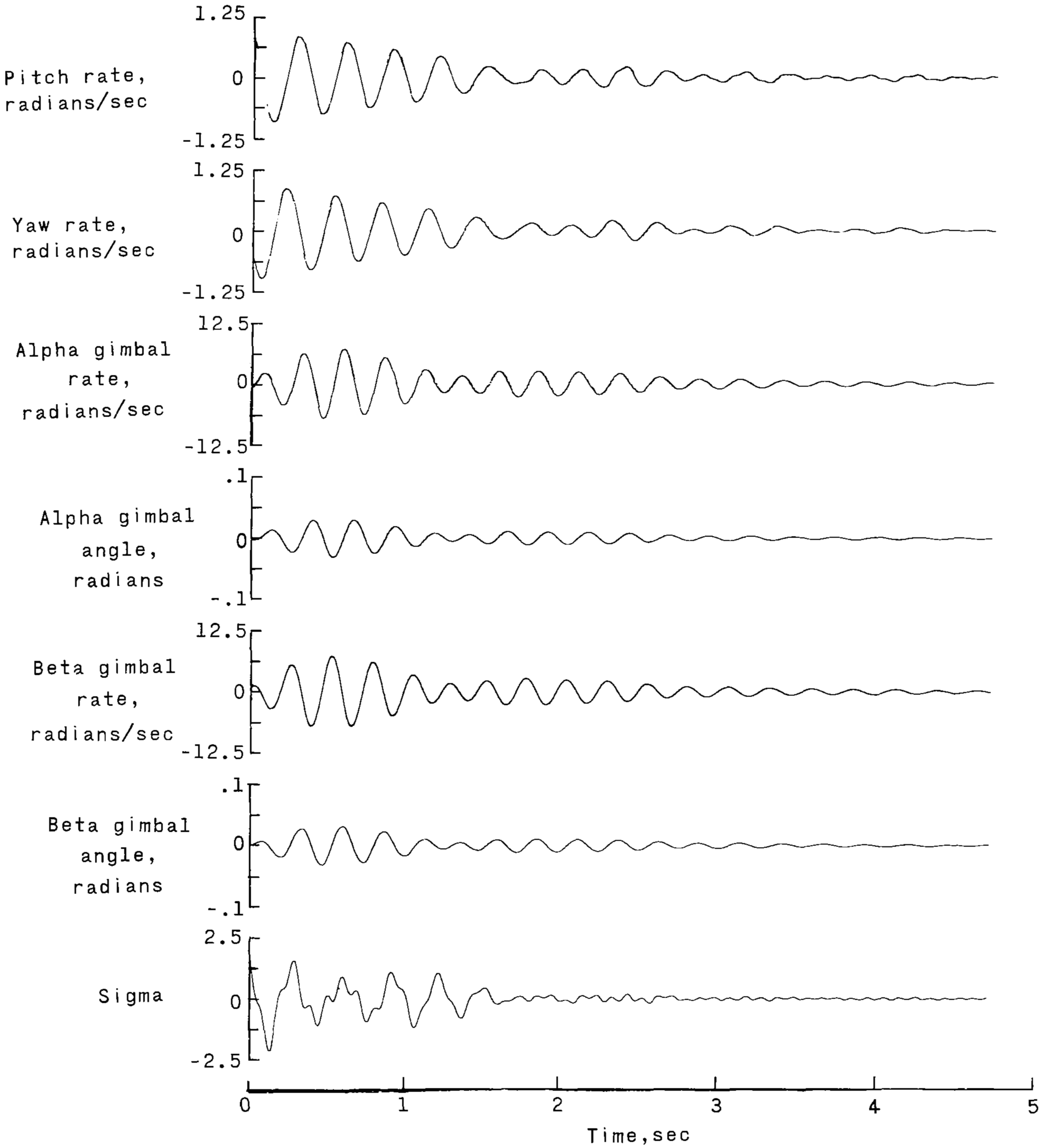
In order to evaluate the utility of the system for nulling the effects of external torques such as those due to thrust misalinement, constant torques as large as 75 percent of the available control torque were simulated. Since the control was asymmetric (pitch torques only), torques effective in the pitch plane, in the yaw plane, and inclined 45° to both planes were tested. Time histories for these three conditions are shown in figure 6. All these responses were for an external torque magnitude equal to 75 percent of the control torque. The damping for these cases was not substantially changed from the nominal case response.

An apparent difficulty that might be encountered by this system is a dependence of performance on the frequency at which the switching hyperplane was oscillated. To investigate this possibility, the spin rate of the simulated vehicle was increased as much as 50 percent and decreased as much as 50 percent while the switching hyperplane was oscillated with the original frequency. Time histories of the extreme cases (±50 percent variation in vehicle spin rate) are shown in figures 7 and 8. Figures 7 and 8 show the high $\left(p_o = 37.5 \text{ radians/sec}\right)$ and low $\left(p_o = 12.5 \text{ radians/sec}\right)$ spin rates, respectively. It is apparent that the effect of off-nominal vehicle spin rate is not significantly deleterious to system performance.



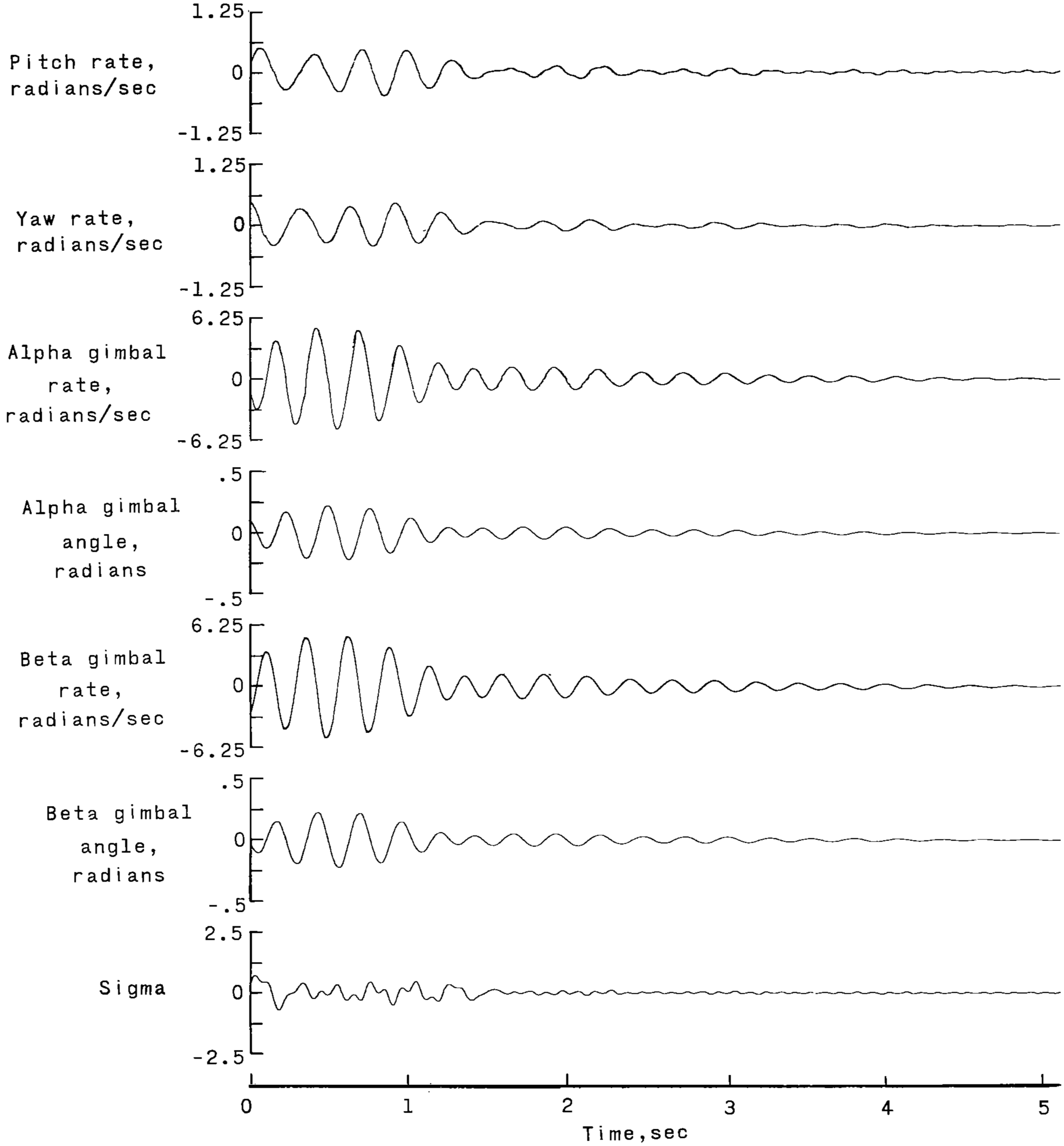
(a) Initial angle $\beta = 0.2$ radian.

Figure 5.- Time history of nominal system response for $\hat{b}_1 = -1.5$ and $\hat{b}_4 = 2.5$.



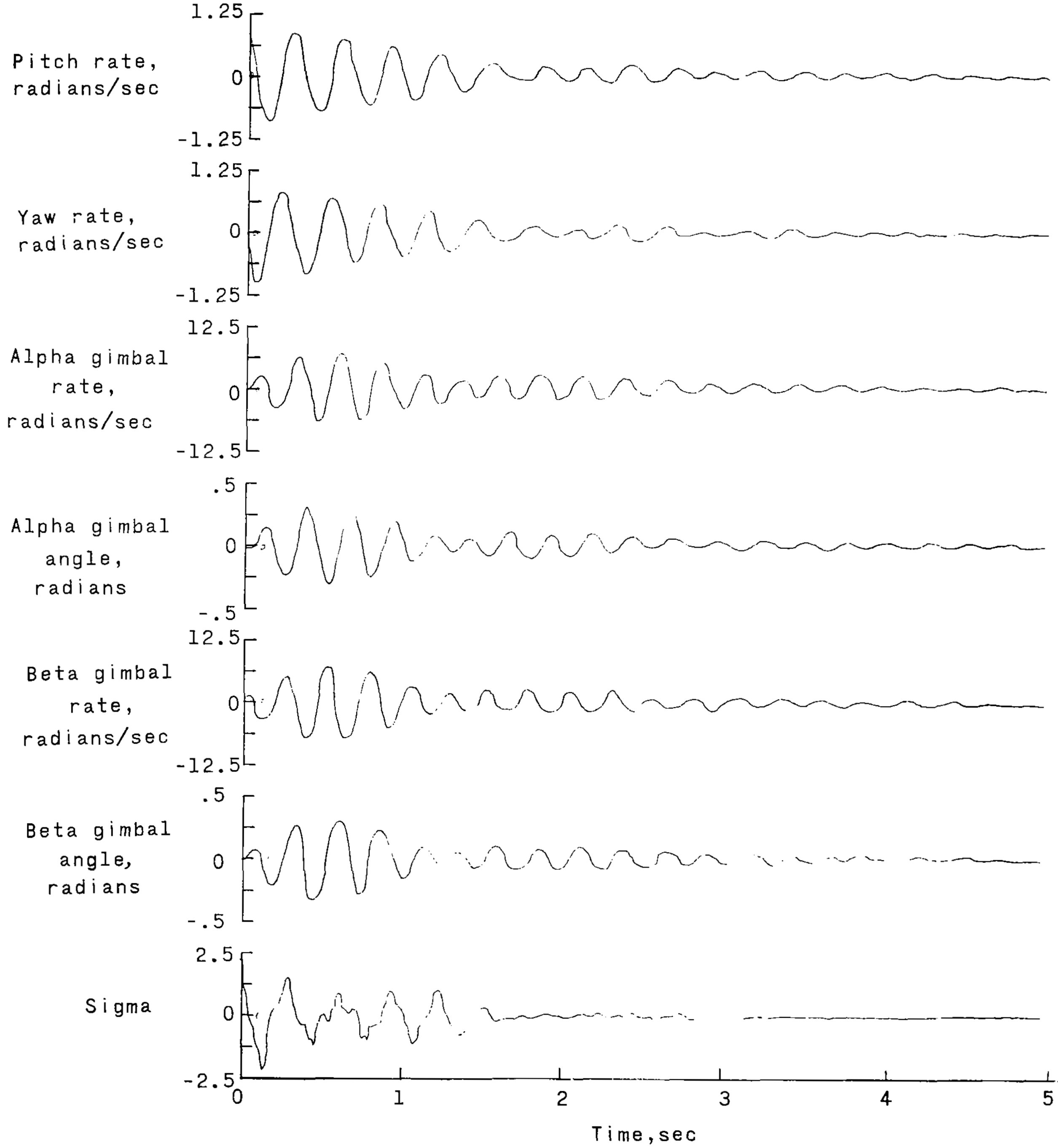
(b) Initial angular rate q = 1.0 radian/sec.

Figure 5.- Continued.



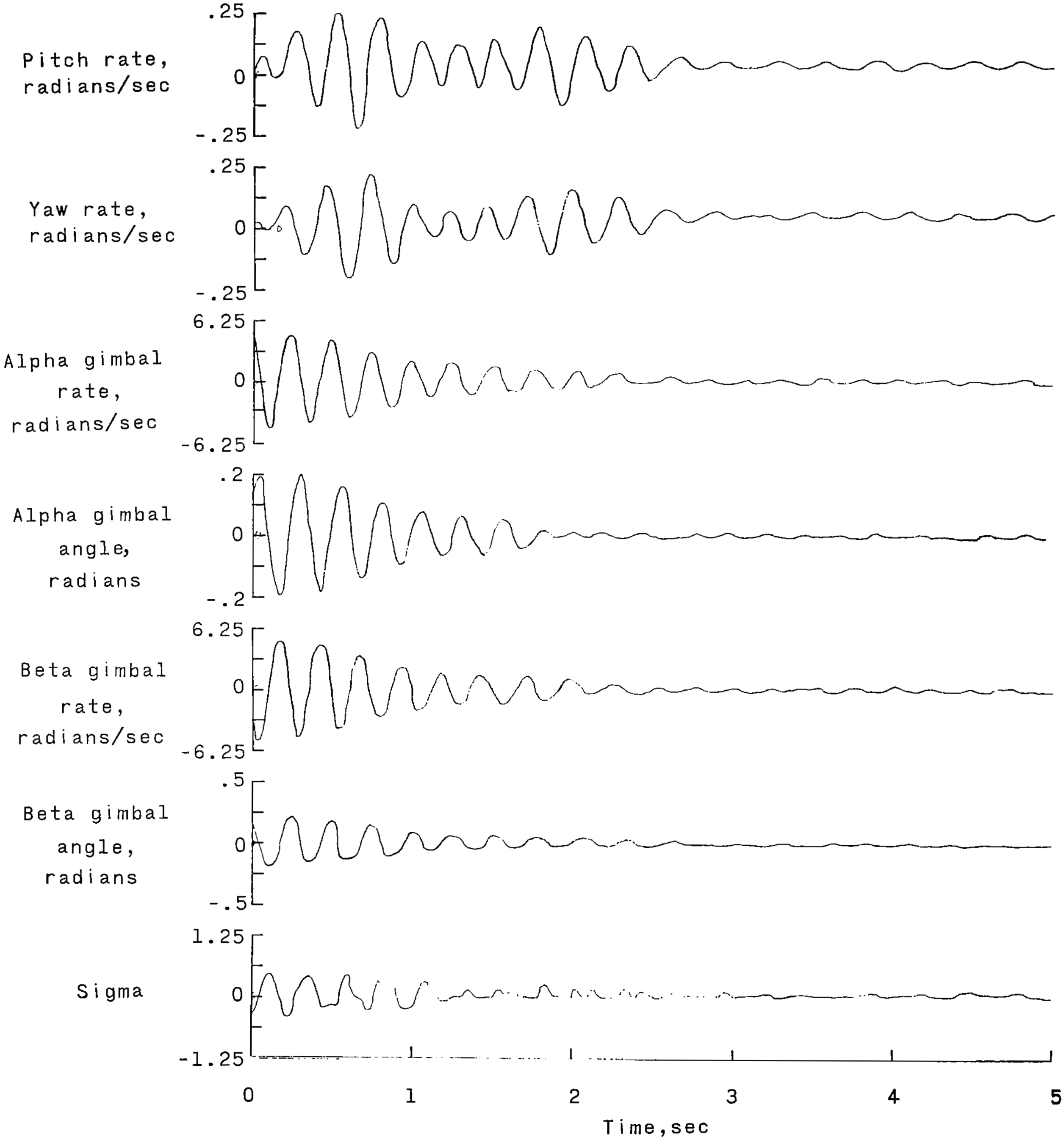
(c) Combined initial conditions $r = 0.5$ radian/sec and $\alpha = 0.1$ radian.

Figure 5.- Concluded.



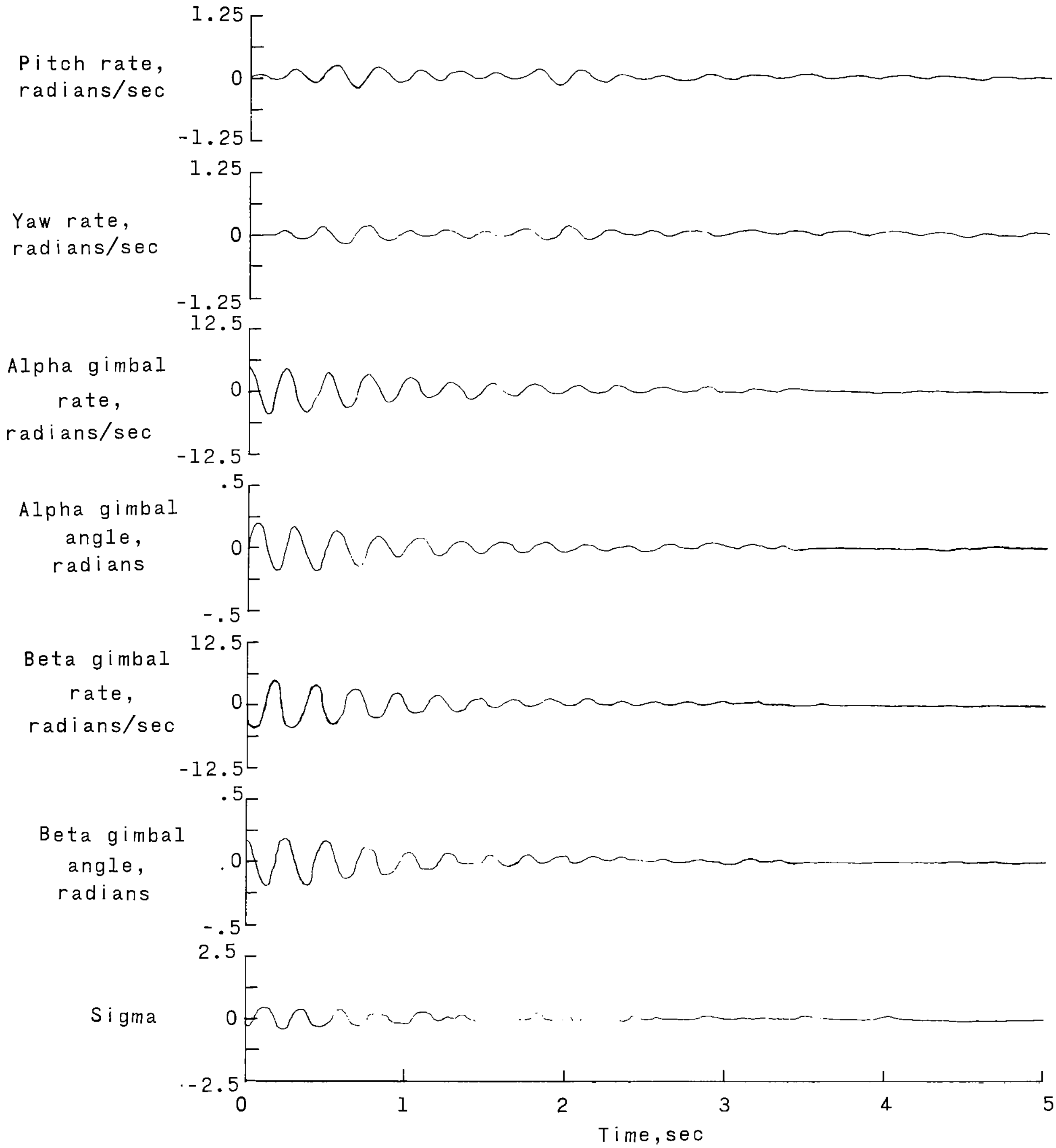
(a) Initial angular rate q = 1.0 radian/sec and external pitching moment equal to 0.75J1.

Figure 6.- Time history of nominal system response.



(b) Initial angle $\beta = 0.2$ radian and external yawing moment equal to 0.75J1.

Figure 6.- Continued.



(c) Initial angle $\beta = 0.2$ radian and external moment equal to 0.75J1 with equal pitch and yaw components.

Figure 6.- Concluded.

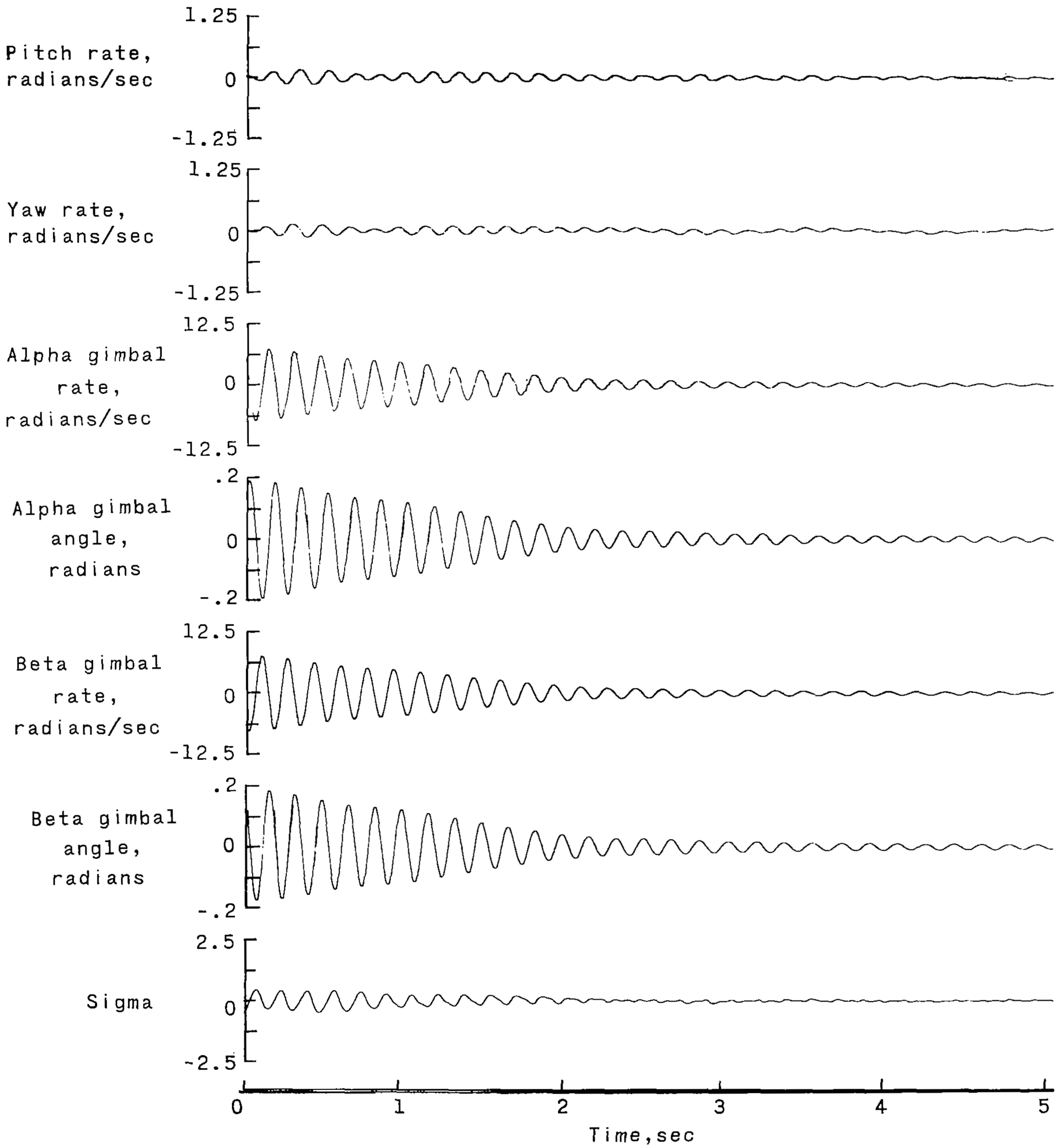


Figure 7.- Time history of system with vehicle spin rate increased 50 percent, nominal feedback frequency, and initial angle $\beta = 0.2$ radian.

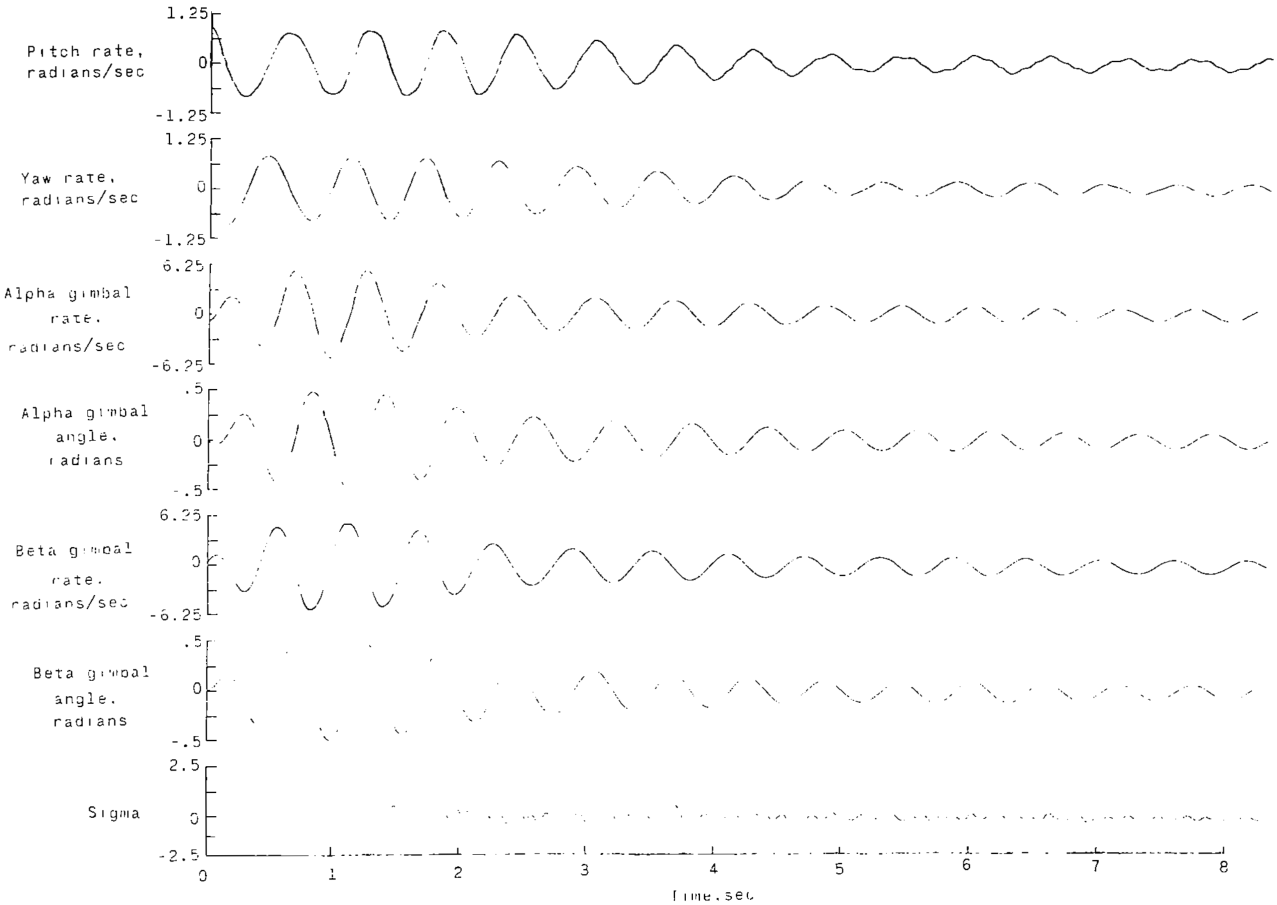
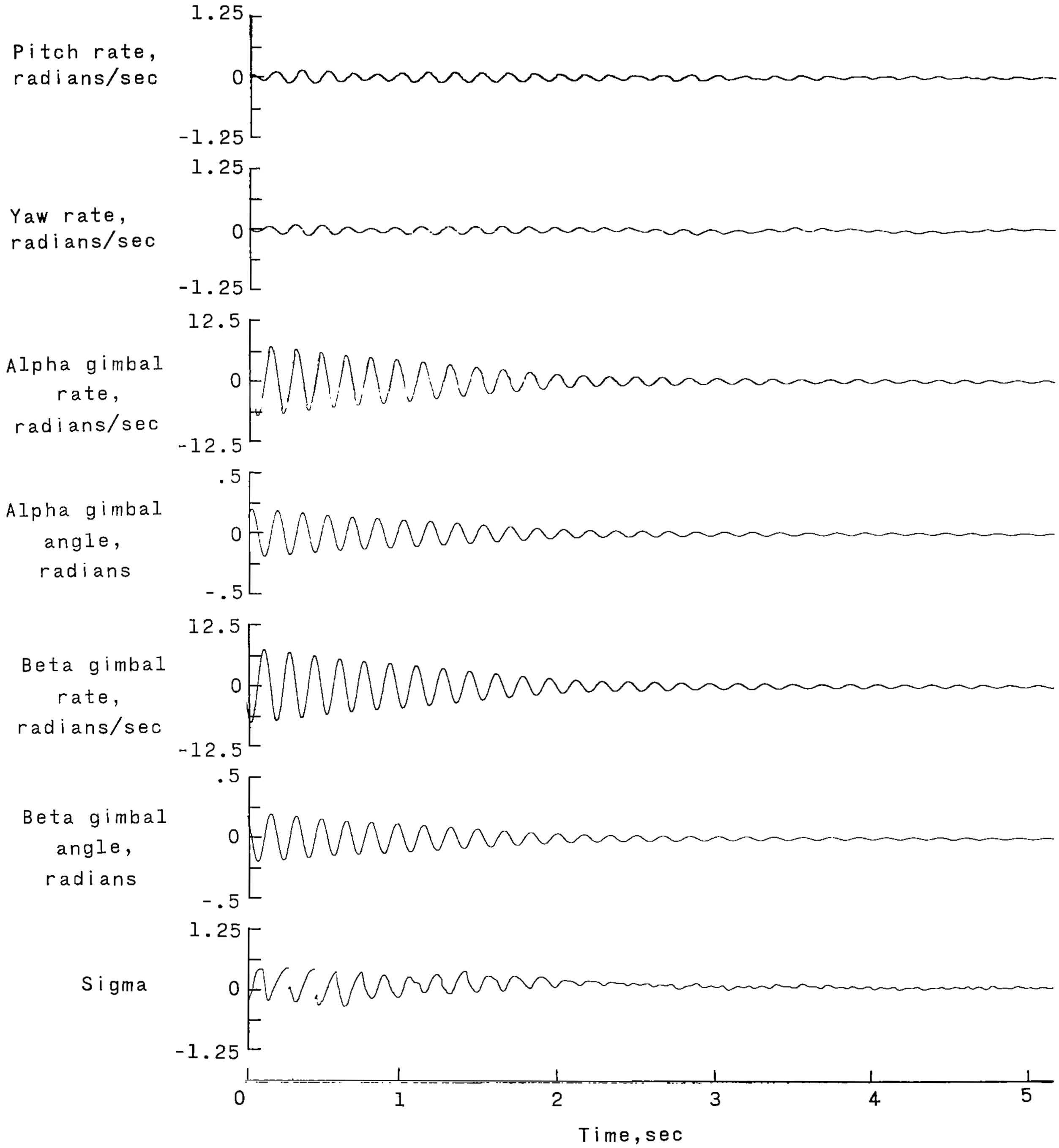


Figure 8.- Time history of system with vehicle spin rate reduced 50 percent, nominal feedback frequency, and initial angular rate $q = 1.0$ radian/sec.

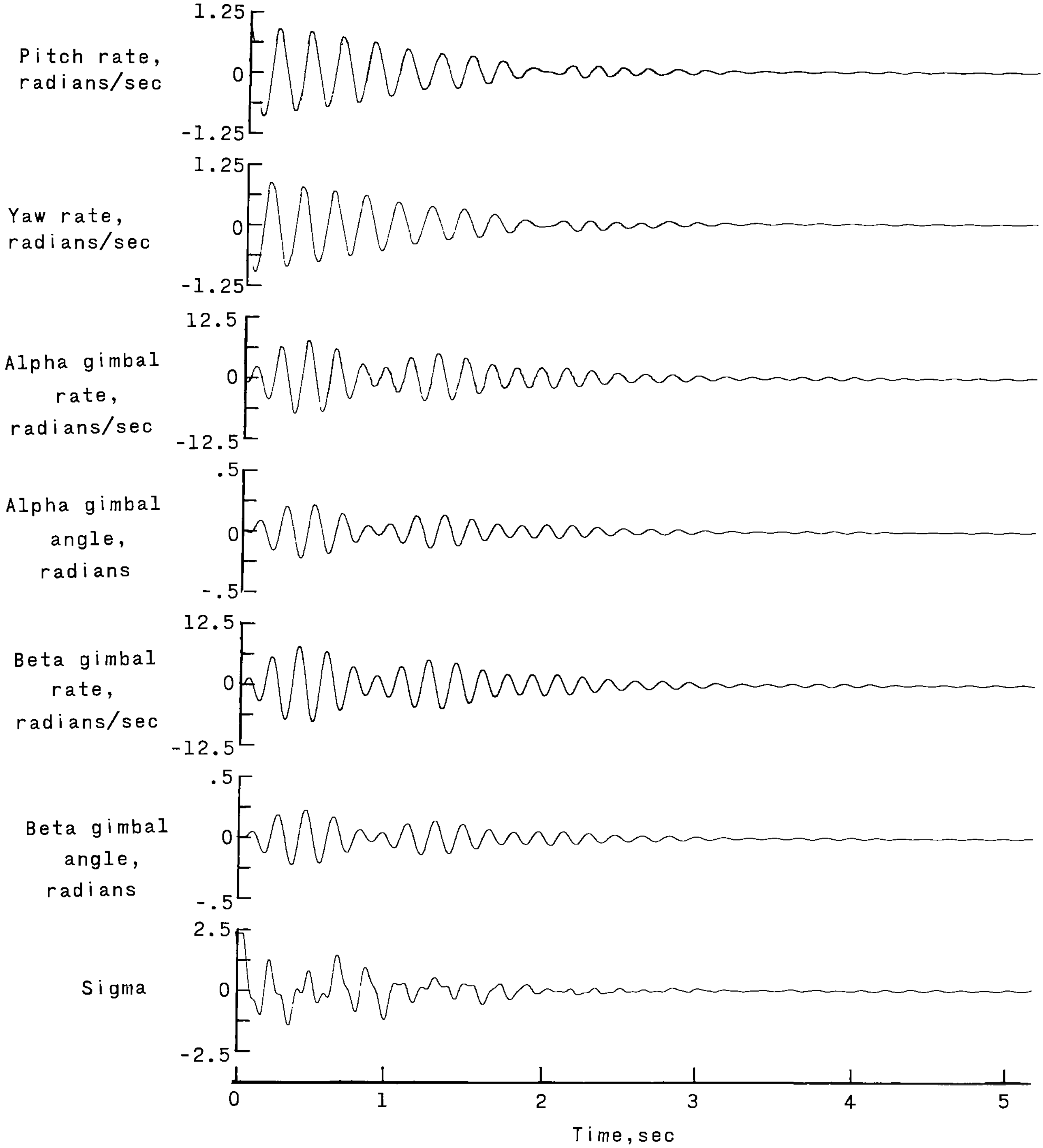
Cases where an interference torque was present and the roll was also off nominal were run. The high roll rate case for pitch and yaw interference torques is represented by figure 9. Similarly, the low roll rate case is represented in figure 10. The limit cycle behavior in these cases correlates well with the nominal roll rate case.

Finally, a time optimal control case was calculated on the digital computer. The nonoptimal control system was then calculated from the same initial conditions. Time histories of $\left\|\vec{x}(t)\right\|$ comparing the responses of the two systems are shown in figure 11. Note that the nonoptimal (but closed loop) logic yields comparable damping, even though the number of switches was about four times that of the optimal system for this set of initial conditions. This comparison is intended to show that the system with linear switching is basically efficient, and it is already apparent that this type of logic is capable of handling a variety of random inputs and off-nominal system characteristics.



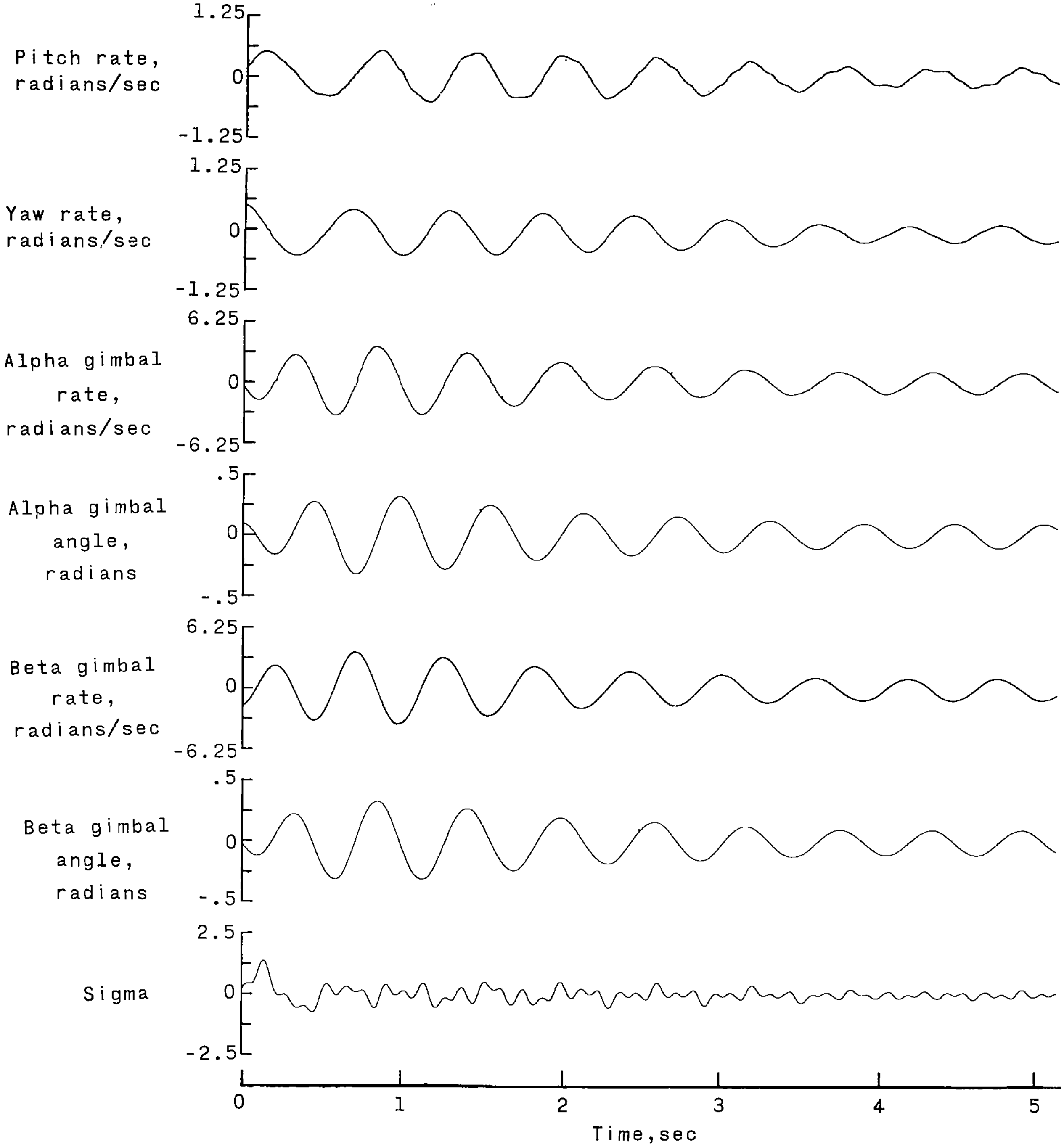
(a) Initial angle $\beta = 0.2$ radian, and external pitching moment equal to 0.75J1.

Figure 9.- Time history of system with vehicle spin rate increased 50 percent, and nominal feedback frequency.



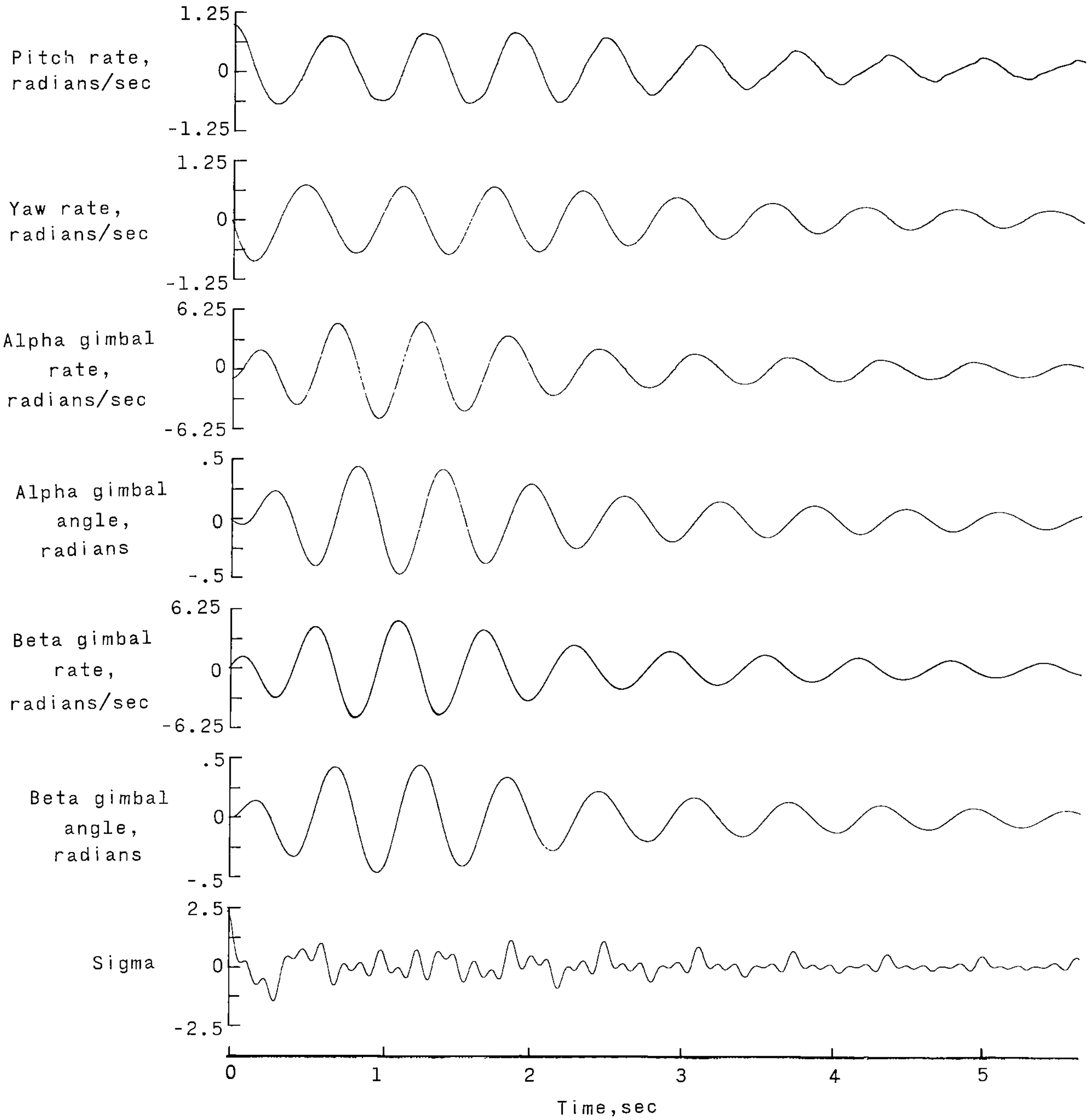
(b) Initial angular rate $q = 1.0$ radian/sec and external yawing moment equal to 0.75J1.

Figure 9.- Concluded.



(a) Combined initial conditions $\alpha = 0.1$ radian, $r = 0.5$ radian/sec, and external pitching moment equal to 0.75J1.

Figure 10.- Time history of system with vehicle spin rate reduced 50 percent, and nominal feedback frequency.



(b) Initial angular rate $q = 1.0$ radian/sec and external yawing moment equal to 0.75J1.

Figure 10.- Concluded.

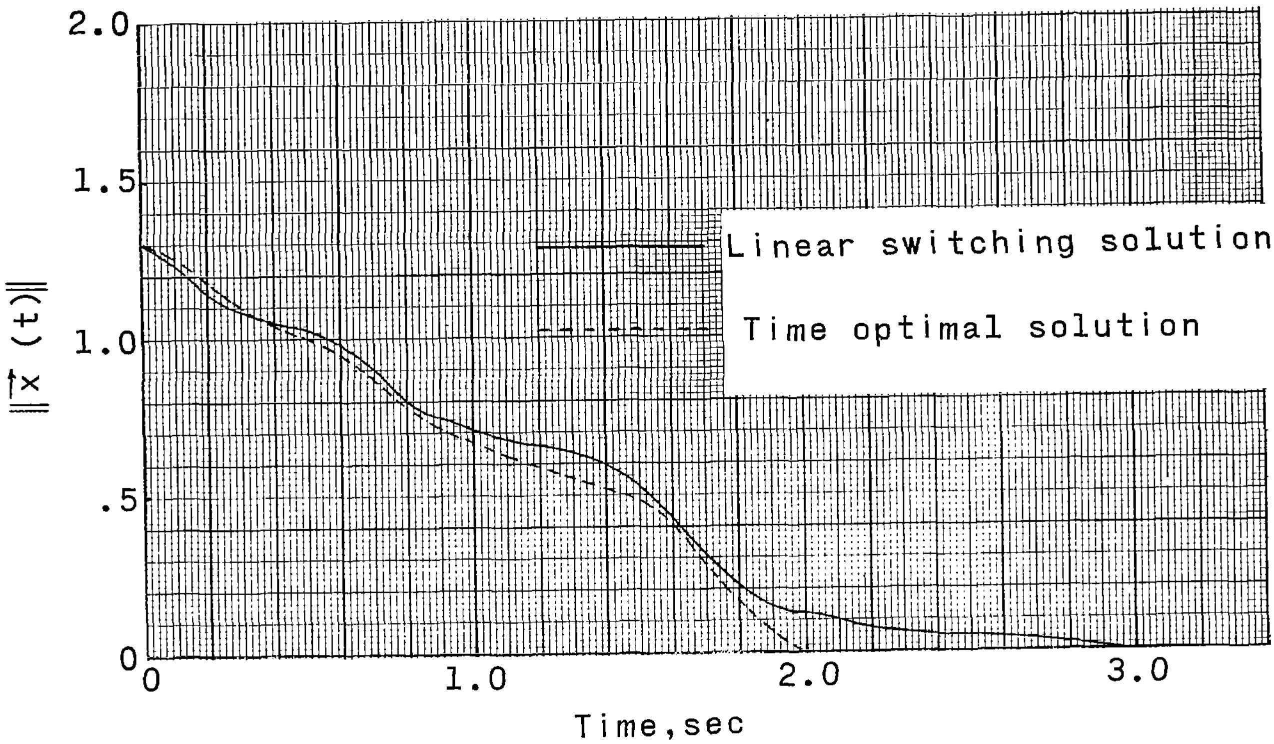


Figure 11.- Time histories of $\|\vec{x}(t)\|$ comparing the response of the system with closed-loop linear switching with the response of the time optimal system for same initial conditions. $q_o = 0.734$ radian/sec; $r_o = -1.02$ radians/sec; $\alpha_o = 0.248$ radian; and $\beta_o = 0.117$ radian.

CONCLUDING REMARKS

The intent of this paper has been to present and discuss a technique for the synthesis of a particular class of bang-bang control systems. This technique, which uses linear switching with time-dependent gains, was applied to an example problem and the stability and flexibility of the control under a variety of conditions has been demonstrated.

This control-synthesis technique appears to be applicable to a wide range of control problems and offers a method for alleviating the endpoint problem which is common to linear switching with constant gains. Future developments of this technique should include generalization to systems with more than one control and considerations of switching delay times, hysteresis, and transport lags.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., December 5, 1963.

APPENDIX

DYNAMIC AND CONTROL LOGIC EQUATIONS

Vehicle Equations

The equations which were used to describe the dynamics of the vehicle were Euler's dynamic equations. (See ref. 15.) The following assumptions were employed in the description:

(1) No coupling between the force and moment equations

(2) Symmetric inertia distribution, that is, $I_Y = I_Z = I$

(3) No X-component of torque so that $p = p_o > 0$

(4) The moments of inertia satisfy the inequality $I_X < I$.

With these assumptions, the equations were written as

$$\dot{q} - \omega r = Ju(t) + \frac{M_Y}{I} \tag{A1}$$

$$\dot{r} + \omega q = \frac{M_Z}{I} \tag{A2}$$

where

$$\omega = \left(1 - \frac{I_X}{I}\right)p_o \tag{A3}$$

$$\left|u(t)\right| \leqq 1 \tag{A4}$$

and IJ is the maximum available control torque.

General Gimbal Angle Equations

In deriving the equations for α and β, the following assumptions were made:

(1) No torques act on the free gyro

(2) The spin axis and total angular momentum vector of the free gyro are coincident and alined with a reference direction in inertial space.

With these assumptions, a unit vector $\vec{e}$ in the direction of the free gyro spin axis satisfies the following vector differential equation written in the principal vehicle axis system:

$$\frac{d}{dt}(\vec{e}) + \vec{\Omega} x \vec{e} = 0 \tag{A5}$$

where

$$\vec{\Omega} = p_o\vec{i} + q\vec{j} + r\vec{k} \tag{A6}$$

and from figure 3 it is easily seen that

$$\vec{e} = \cos\alpha \cos\beta\vec{i} + \sin\beta\vec{j} - \sin\alpha \cos\beta\vec{k} \tag{A7}$$

Linearized Gimbal Angle Equations

The scalar equations corresponding to equation (A5) were linearized by making the following assumptions:

(1)

$$\left.\begin{aligned} \sin\alpha &= \alpha \\ \cos\alpha &= 1 \\ \sin\beta &= \beta \\ \cos\beta &= 1 \end{aligned}\right\} \tag{A8}$$

(2) The products $\alpha\dot{\alpha}$, $\beta\dot{\beta}$, $q\alpha$, and $r\beta$ are small quantities and may be neglected. Under these assumptions, the differential equations for α and β are:

$$\dot{\alpha} - p_o\beta = -q \tag{A9}$$

$$\dot{\beta} + p_o\alpha = -r \tag{A10}$$

State Vector Form of Equations

In order to express equations (A1), (A2), (A9), and (A10) in state vector form, define

$$\vec{x} = \begin{bmatrix} q \\ r \\ \alpha \\ \beta \end{bmatrix} \tag{A11}$$

$$[A] = \begin{bmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ -1 & 0 & 0 & p_o \\ 0 & -1 & -p_o & 0 \end{bmatrix} \tag{A12}$$

$$\vec{a} = \begin{bmatrix} J \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (J > 0) \tag{A13}$$

Then for $M_Y = M_Z = 0$, the equations can be written as

$$\dot{\vec{x}} = [A]\vec{x} + u\vec{a} \tag{A14}$$

Note that the controllability condition is satisfied since

$$\det\left(\vec{a}, A\vec{a}, A^2\vec{a}, A^3\vec{a}\right) = J^4\omega p_o(\omega + p_o)^2 \neq 0 \tag{A15}$$

Control Logic Equations

For this problem, the control law has the form

$$u(\vec{x};t) = \mathrm{sgn}\left[b_1(t)q + b_2(t)r + b_3(t)\alpha + b_4(t)\beta\right] \tag{A16}$$

where $\vec{b}$ has the form $-[B]\vec{a}$ and all the roots of

$$\det\left([A] + \vec{a}\vec{b}^* - \lambda[I]\right) = 0 \tag{A17}$$

have negative real parts. If $\vec{b}$ is assumed to have the form

$$\vec{b} = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ b_4 \end{bmatrix} \tag{A18}$$

equation (A17) can be written as

$$\lambda^4 - Jb_1\lambda^3 + \left(p_o^{\ 2} + \omega^2\right)\lambda^2 - J\left[b_1 p_o^{\ 2} + \left(p_o + \omega\right)b_4\right]\lambda + \omega^2 p_o^{\ 2} = 0 \qquad \text{(A19)}$$

Application of the Routh-Hurwitz stability criterion (see, for example, ref. 16) to equation (A19) shows that the roots will have negative real parts if

$$0 < b_4 < -\frac{b_1 I_X p_o}{I} \qquad \text{(A20)}$$

If $\vec{b}(t)$ has the form

$$\vec{b}(t) = \begin{bmatrix} \hat{b}_1 - \cos 2\omega t \\ 0 \\ 0 \\ \hat{b}_4 \end{bmatrix} \qquad \text{(A21)}$$

it is easy to find values for the constants $\hat{b}_1$ and $\hat{b}_4$ such that inequality (A20) holds, and if any value of $t \geqq 0$ is given, there exists a $B = B^* > 0$ such that $\vec{b} = -B\vec{a}$.

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